Quantization conditions from the group theory

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Abstract

We show that the requirement of the relativistic invariance for any self-interacting, abelian p-form theory uniquely determines the form of the corresponding quantization condition.

The old idea of electric-magnetic duality has played in recent years very prominent role in quantum field theory and string theory (see e.g. [1]). One of its most fascinating implication is the celebrated Dirac quantization condition [2]

$$eg = nh \ , \tag{1}$$

with integer n (h is the Planck constant). This idea may be in a straightforward manner generalized to self-interacting p-form electrodynamics in D dimensions [3]–[6]. Then the condition (1) still holds with e and g being the electric and magnetic charges of the elementary electric $(p-1)$–brane and magnetic $(D-p-3)$–brane.

When $D = 2p + 2$, $(p-1)$–brane dyons may exist. In $D = 4$ (i.e. for $p = 1$) it was shown by Zwanziger and Schwinger [7] that the Dirac condition (1) should be replaced by

$$e_1 g_2 - e_2 g_1 = nh \ . \tag{2}$$

Now, for $p > 1$, the quantization condition for $(p-1)$–brane dyons crucially depend upon the parity of p [8]. When p is odd it is simply a generalization of (2) with e and g being the electric and magnetic charges of $(p-1)$–brane dyon. But when p is even one has to replace (1) by

$$e_1 g_2 + e_2 g_1 = nh \ . \tag{3}$$

In [8] both conditions were derived by introducing the Dirac p-branes (the generalization of Dirac strings [2]) and taking into account the multiple connectedness of the configuration space of Dirac p-branes and $(p-1)$–brane dyons. It should be stressed

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that both conditions are valid for any self-interacting (not only Maxwell one), gauge-invariant electrodynamics irrespective of its duality invariance.

In the present paper we show that knowing the Dirac condition (1), both conditions (2) and (3) follow immediately from the group theory. Our argument goes as follows: note first, that the Dirac condition (1) may be generalized only either to (2) or (3). There is no other possibility. The main difference between (2) and (3) lies in the corresponding symmetry groups. Note, that both conditions are invariant under the following scaling transformations:

$$
\begin{aligned}
e & \rightarrow \lambda e \, , \\
g & \rightarrow \frac{1}{\lambda} g \, ,
\end{aligned}
\tag{4}
$$

with $\lambda \neq 0$. But condition (2), contrary to (3), is additionally invariant under $SO(2)$ orthogonal rotations

$$
\begin{aligned}
e & \rightarrow e \cos \alpha - g \sin \alpha \, , \\
g & \rightarrow e \sin \alpha + g \cos \alpha \, ,
\end{aligned}
\tag{5}
$$

and $SO(1,1)$ hyperbolic rotations

$$
\begin{aligned}
e & \rightarrow e \cosh \alpha + g \sinh \alpha \, , \\
g & \rightarrow e \sinh \alpha + g \cosh \alpha \, .
\end{aligned}
\tag{6}
$$

Note, that transformations (4)–(6) generate $SO(2,1)$ group, i.e. they realize the $SO(2,1)$ group as linear transformations in the 2-dimensional space parameterized by (e,g). Now, condition (3) is neither invariant under (5) nor under (6), but it is invariant under the discrete Z_2 transformation

$$
e \rightarrow g \, , \qquad g \rightarrow e \, .
\tag{7}
$$

Obviously, condition (2) is not Z_2 invariant. Since (4) defines the $SO(1,1)$ group (subgroup of $S(2,1)$), therefore, condition (2) is $SO(2,1)$ invariant, whereas condition (3) is $SO(1,1) \times Z_2$ invariant. Note, that (5) are nothing but the duality rotations between electric and magnetic charges.

Now, it turns out that the same symmetry groups are encoded into the canonical structure of any gauge-invariant electrodynamics. Consider first $p = 1$ theory in $D = 4$. It is easy to see that the canonical Poisson bracket

$$
\{D^i(x), B^j(y)\} = \epsilon^{ikj} \partial_k \delta^{(3)}(x - y) \, ,
\tag{8}
$$

is invariant under the same group of transformations (4)–(5) with (e,g) replaced by $(\mathbf{D}, \mathbf{B})$. Physically, the $SO(2,1)$ group is implied be the relativistic invariance. In the

Hamiltonian framework this invariance is equivalent to the symmetry of the corresponding energy-momentum tensor. Now, any Hamiltonian is a functional of the following three scalar functions built out of $\mathbf{D}$ and $\mathbf{B}$ [9] (see [10] for a p-form generalization):

$$\alpha = \frac{1}{2}(\mathbf{D}^2 + \mathbf{B}^2) , \qquad (9)$$

$$\beta = \frac{1}{2}(\mathbf{D}^2 - \mathbf{B}^2) , \qquad (10)$$

$$\gamma = \mathbf{DB} . \qquad (11)$$

The symmetry of $T^{\mu\nu}$ implies the following equation for the Hamiltonian $\mathcal{H}$:

$$(\partial_\alpha \mathcal{H})^2 - (\partial_\beta \mathcal{H})^2 - (\partial_\gamma \mathcal{H})^2 = 1 , \qquad (12)$$

which displays exactly the $SO(2,1)$ symmetry. Therefore, transformations (4)–(5) with (e, g) replaced by $(\mathbf{D}, \mathbf{B})$ are nothing but a realization in the space of fields $(\mathbf{D}, \mathbf{B})$ of this *fundamental* symmetry. Summarizing: the invariance group of the standard Dirac-Zwanziger-Schwinger condition (2) is implied by the relativistic invariance of the underlying (possibly nonlinear) electrodynamics.

Now, we demand that the same property holds for any p-form theory, i.e. that the symmetry group of the corresponding quantization condition is implied by the relativistic invariance of the corresponding p-form theory. When p is odd any Hamiltonian is built out of α, β and γ, where now $\mathbf{XY} = \frac{1}{p!} X^{i_1 \ldots i_p} Y_{i_1 \ldots i_p}$ for any p-forms X and Y. Therefore, all arguments are the same as for $p = 1$ and the corresponding quantization condition is given by (2). When p is even the situation is different. Now $\mathbf{DB} = 0$ and, therefore, any Hamiltonian depends only upon α and β. The symmetry of $T^{\mu\nu}$ gives rise to

$$(\partial_\alpha \mathcal{H})^2 - (\partial_\beta \mathcal{H})^2 = 1 , \qquad (13)$$

which, contrary to (12), displays only the $SO(1,1)$ symmetry. This group is realized in the space of fields (p-forms) (D, B) as the group of canonical transformations. It is easy to see that the canonical Poisson bracket (a p-form generalization of (8))

$$\{D^{i_1 \ldots i_p}(x), B^{j_1 \ldots j_p}(y)\} = \epsilon^{i_1 \ldots i_p k j_1 \ldots j_p} \partial_k \delta^{(2p+1)}(x - y) , \qquad (14)$$

is (for even p) no longer invariant under the full $SO(2,1)$ group but only under the $SO(1,1)$ subgroup generated by (4) (with (e, g) replaced by (D, B)). Moreover, since for any p-form theory

$$T^{0k} = \frac{1}{p!}(-1)^{p+1} \epsilon^{k i_1 \ldots i_p j_1 \ldots j_p} D_{i_1 \ldots i_p} B_{j_1 \ldots j_p} , \qquad (15)$$

$$T^{k0} = \frac{1}{p!}(-1)^{p+1} \epsilon^{k i_1 \ldots i_p j_1 \ldots j_p} E_{i_1 \ldots i_p} H_{j_1 \ldots j_p} , \qquad (16)$$

therefore, for even p the condition $T^{0k} = T^{k0}$ is obviously invariant under the discrete Z_2 transformation:

$$D \to B , \qquad B \to D . \qquad (17)$$

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Note, that (14) is also invariant under (17), i.e. (17) defines a canonical symmetry.

Summarizing: the relativistic invariance of any p-form theory defines the $SO(2,1)$ and $SO(1,1) \times Z_2$ groups of canonical symmetries for odd and even p respectively, i.e. both canonical structure and quantization condition have the same symmetry properties. Therefore, the structure of the corresponding quantization condition ((2) or (3)) is uniquely determined by the requirement of the relativistic invariance of the underlying p-form theory.

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